A new extended discrete KP hierarchy and a generalized dressing method

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Abstract
Inspired by the squared eigenfunction symmetry constraint, we introduce a new τ_k-flow by 'extending' a specific t_n-flow of a discrete KP hierarchy (DKPH). We construct an extended discrete KPH (exDKPH), which consists of t_n-flow, τ_k-flow and t_n evolution of eigenfunction and adjoint eigenfunctions, and its Lax representation. The exDKPH contains two types of discrete KP equation with self-consistent sources (DKPESCS). Two reductions of exDKPH are obtained. The generalized dressing approach for solving the exDKPH is proposed and the N-soliton solutions of two types of the DKPESCS are presented.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

Generalizations of a soliton hierarchy attract a lot of interests from both physical and mathematical points and there were some methods to generalize the soliton hierarchy [1–4]. Recently, a systematic approach inspired by a squared eigenfunction symmetry constraint was proposed to construct the extended KP hierarchy [5]. By this method, the extended two-dimensional Toda lattice hierarchy, the extended CKP hierarchy and the extended q-deformed KP hierarchy have been obtained [6–8].

The discrete KP hierarchy(DKPH) [9–12] is an interesting object in the research of the discrete integrable systems and the discretization of the integrable systems [13]. Sato's approach for the discrete KPH was presented in [11]. Naturally, there are some similar properties between discrete KPH and KPH [14], such as tau function [12, 14], Hamiltonian structure [12] and gauge transformation [10, 15, 16], etc. In [10], Oevel has explicitly given

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two types of gauge transformation operators of the discrete KPH. In [16], the combined gauge operator and the determinant representation of the operator have been obtained.

In this paper, we will construct the extension of the discrete KPH (exDKPH). Inspired by the squared eigenfunction symmetry constraint of the discrete KP hierarchy [10], we introduced a new τ_k-flow by 'extending' a specific t_n-flow of the discrete KP hierarchy. Then we find the exDKPH consisting of the t_n-flow of the discrete KP hierarchy, τ_k-flow and the t_n-evolutions of eigenfunctions and adjoint eigenfunctions. The commutativity of t_n-flow and τ_k-flow gives rise to zero curvature representation for exDKPH. Also the Lax representation of exDKPH is derived. Due to the introduction of τ_k-flow the exDKPH contains two time series $\{t_n\}$ and $\{\tau_k\}$ and more components by adding eigenfunctions and adjoint eigenfunctions. The exDKPH contains the first and second types of the discrete KP equation with self-consistent sources (DKPESCS). The KP equation with self-consistent sources arose in some physical models describing the interaction of long and short waves [4]. The similarity between the KP equation and the discrete KP equation enables us to speculate on the potential application of the discrete KP equation with self-consistent sources. By t_n-reduction and τ_k-reduction, the exDKPH reduces to a discrete (1+1)-dimensional integrable hierarchy with self-consistent sources and a constrained discrete KP hierarchy, respectively.

The dressing method is an important tool for solving the soliton hierarchy [12]. However, this method cannot be applied directly for solving the 'extended' hierarchy. A generalized dressing approach for exKPH is proposed in [17]. In this paper, with the combination of the dressing method and variations in the constants method, a generalization to the dressing method for exDKPH is presented, which is based on the dressing method for discrete KPH [11] and a similar approach for finding Wronskian solutions to the constrained KP hierarchy [18]. In this way, we can solve the entire hierarchy of exDKPH in an unified and simple manner. As the special cases, the N-soliton solutions of *both* types of DKPESCS are obtained simultaneously.

This paper will be organized as follows. In section 2, we present the exDKPH and its Lax pair, which includes the two types of DKPESCS. In section 3, t_n-reduction and τ_k-reduction for the exDKPH are given. In section 4, we discuss the generalized dressing method for the exDKPH. In section 5, we present the N-soliton solutions of the DKPESCS.

2. New extended discrete KP hierarchy

We denote the shift and the difference operators acting on the associative ring F of functions by Γ and Δ, respectively, as follows:

$$F = \{f(l) = f(l, t_1, t_2, \ldots, t_i, \ldots); l \in \mathbb{Z}, t_i \in \mathbb{R}\}$$
$$\Gamma(f(l)) = f(l+1) = f^{(1)}(l), \qquad \Delta(f(l)) = f(l+1) - f(l).$$

In this paper, we use $P(f)$ to denote an action of the difference operator P on the function f, while Pf means the multiplication of the difference operator P and the zero-order difference operator f. Define the following operation

$$\Delta^j f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i (f(l+j-i))) \Delta^{j-i},$$

$$\binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{1}$$

Also, we define the adjoint operator to the Δ operator by Δ^*:

$$\Delta^*(f(l)) = (\Gamma^{-1} - I)(f(l)) = f(l-1) - f(l), \tag{2}$$

$$\Delta^{*j} f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^{*i}(f(l+i-j)))\Delta^{*j-i}. \tag{3}$$

Let $P = \sum_{j=-\infty}^{k} f_j(l)\Delta^j$, the adjoint operator P^* is defined by $P^* = \sum_{j=-\infty}^{k} \Delta^{*j} f_j(l)$.

The Lax equation of the DKP hierarchy is given by [9, 11]

$$L_{t_n} = [B_n, L], \tag{4}$$

where $L = \Delta + f_0 + f_1\Delta^{-1} + f_2\Delta^{-2} + \cdots$ is a pseudo-difference operator with potential functions $f_i \in F$, $B_n = L_+^n$ stands for the difference part of L^n. The commutativity of t_n- and t_m-flow gives rise to the zero-curvature equations for the DKP hierarchy:

$$B_{n,t_m} - B_{m,t_n} + [B_n, B_m] = 0 \tag{5}$$

with the Lax pair given by

$$\psi_{t_n} = B_n(\psi), \qquad \psi_{t_m} = B_m(\psi). \tag{6}$$

The t_n evolutions of eigenfunction ψ and adjoint eigenfunction ϕ read

$$\psi_{t_n} = B_n(\psi), \qquad \phi_{t_n} = -B_n^*(\phi). \tag{7}$$

For $n = 2, m = 1$, (5) gives rise to the DKP equation [9]

$$\Delta(f_{0t_2} + 2f_{0t_1} - 2f_0 f_{0t_1}) = (\Delta + 2) f_{0t_1 t_1}. \tag{8}$$

It is known that the squared eigenfunction symmetry constraint given by [10]

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i$$

$$\psi_{i,t_n} = B_n(\psi_i), \qquad \phi_{i,t_n} = -B_n^*(\phi_i), \qquad i = 1, \ldots, N,$$

is compatible with the DKP hierarchy. Here N is an arbitrary natural number, ψ_i and ϕ_i are N different eigenfunctions and adjoint eigenfunctions of equations (9c). This compatibility enables us to construct a new extended discrete KP hierarchy (exDKPH) as

$$L_{t_n} = [B_n, L], \tag{9a}$$

$$L_{\tau_k} = \left[B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i, L \right], \tag{9b}$$

$$\psi_{i,t_n} = B_n(\psi_i), \phi_{i,t_n} = -B_n^*(\phi_i), \qquad i = 1, \ldots, N. \tag{9c}$$

We have the following lemma.

Lemma 1. *Let $Q = a\Delta^k, k \geqslant 1$, then*

$$(\Delta^{-1}\phi Q)_- = \Delta^{-1} Q^*(\phi) \tag{10a}$$

$$[B_n, \psi \Delta^{-1} \phi]_- = B_n(\psi)\Delta^{-1}\phi - \psi\Delta^{-1} B_n^*(\phi). \tag{10b}$$

Proof. Using $f\Delta = \Delta\Gamma^{-1}(f) - \Delta(\Gamma^{-1}(f)), \Delta^* = -\Delta\Gamma^{-1}$, we have

$$(\Delta^{-1}\phi a\Delta^k)_- = (\Delta^{-1}\Delta\Gamma^{-1}(\phi a)\Delta^{k-1} - \Delta^{-1}\Delta(\Gamma^{-1}(\phi a))\Delta^{k-1})_-$$
$$= -(\Delta^{-1}\Delta(\Gamma^{-1}(\phi a))\Delta^{k-1})_- = \cdots$$
$$= (-1)^k \Delta^{-1}\Delta^k(\Gamma^{-k}(\phi a)) = \Delta^{-1}\Delta^{*k}(\phi a) = \Delta^{-1}Q^*(\phi)$$

which yields to (10a) and (10b). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 1. *The commutativity of ([9a](#)) and ([9b](#)) under ([9c](#)) gives rise to the following zero-curvature representation for exDKPH (9)*

$$B_{n,\tau_k} - \left(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i\right)_{t_n} + \left[B_n, B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i\right] = 0, \tag{11a}$$

$$\psi_{i,t_n} = B_n(\psi_i), \qquad \phi_{i,t_n} = -B_n^*(\phi_i), \qquad i = 1, 2, \ldots, N, \tag{11b}$$

with the Lax representation given by

$$\Psi_{t_n} = B_n(\Psi), \qquad \Psi_{\tau_k} = \left(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i\right)(\Psi). \tag{12}$$

Proof. For convenience, we omit $\sum$. By (9) and Lemma 1, we have
$$B_{n,\tau_k} = \left(L_{\tau_k}^n\right)_+ = [B_k + \psi \Delta^{-1} \phi, L^n]_+ = \left[B_k + \psi \Delta^{-1} \phi, L_+^n\right]_+ + \left[B_k + \psi \Delta^{-1} \phi, L_-^n\right]_+$$
$$= \left[B_k + \psi \Delta^{-1} \phi, L_+^n\right] - \left[B_k + \psi \Delta^{-1} \phi, L_+^n\right]_- + \left[B_k, L_-^n\right]_+ = \left[B_k + \psi \Delta^{-1} \phi, L_+^n\right]$$
$$- [\psi \Delta^{-1} \phi, B_n]_- + [B_n, L^k]_+ = [B_k + \psi \Delta^{-1} \phi, B_n] + (B_k + \psi \Delta^{-1} \phi)_{t_n}. \qquad \square$$

Remark. The exDKPH (11) extends the DKPH ([5](#)) by containing two time series $\{t_n\}$ and $\{\tau_k\}$ and more components ψ_i and ϕ_i, $i = 1, \ldots, N$.

Example 1. The first type of DKPSCS is given by (11) with $n = 1, k = 2$ [19]

$$\Delta\left(f_{0\tau_2} + 2f_{0t_1} - 2f_0 f_{0t_1}\right) = (\Delta + 2)f_{0t_1 t_1} - \Delta^2 \sum_{i=1}^{N} \left(\psi_i \phi_i^{(-1)}\right), \tag{13a}$$

$$\psi_{i,t_1} = \Delta(\psi_i) + f_0 \psi_i, \qquad \phi_{i,t_1} = -\Delta^*(\phi_i) - f_0 \phi_i, \qquad i = 1, 2, \ldots, N. \tag{13b}$$

Its Lax representation is

$$\Psi_{t_1} = (\Delta + f_0)(\Psi), \tag{14a}$$

$$\Psi_{\tau_2} = \left(\Delta^2 + \left(f_0 + f_0^{(1)}\right)\Delta + \Delta(f_0) + f_1^{(1)} + f_1 + f_0^2 + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i\right)(\Psi). \tag{14b}$$

Example 2. The second type of DKPSCS is given by (11) with $n = 2, k = 1$

$$\Delta\left(f_{0t_2} + 2f_{0\tau_1} - 2f_0 f_{0\tau_1}\right) = (\Delta + 2)f_{0\tau_1 \tau_1} + \sum_{i=1}^{N} \left[\Delta^2\left(\left(f_0 + f_0^{-1} - 2\right)\psi_i \phi_i^{-1}\right)\right.$$
$$\left. + \Delta\left(\psi_i^{(2)} \phi_i - \psi_i \phi_i^{(-2)}\right) + \Delta\left((\Gamma + 1)\left(\psi_i \phi_i^{(-1)}\right)_{\tau_1}\right)\right], \tag{15a}$$

$$\psi_{i,t_2} = \Delta^2(\psi_i) + \left(f_0 + f_0^{(1)}\right)\Delta(\psi_i) + \left(\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2\right)\psi_i, \tag{15b}$$

$$\phi_{i,t_2} = -\Delta^{*2}(\psi_i) - \Delta^*\left(\left(f_0 + f_0^{(1)}\right)\psi_i\right) - \left(\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2\right)\psi_i. \tag{15c}$$

Its Lax representation is

$$\Psi_{t_2} = \left(\Delta^2 + \left(f_0 + f_0^{(1)}\right)\Delta + \Delta(f_0) + f_1^{(1)} + f_1 + f_0^2\right)(\Psi) \tag{16a}$$

$$\Psi_{\tau_1} = \left(\Delta + f_0 + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i\right)(\Psi). \tag{16b}$$

3. Reductions of the exDKPH

3.1. The t_n-reduction

The t_n-reduction is given by

$$L^n = B_n \quad \text{or} \quad L^n_- = 0. \tag{17}$$

Then we have

$$(L^n)_{t_n} = [B_n, L^n] = 0, \qquad B_{n,t_n} = 0.$$

So L is independent of t_n and we have

$$B_n(\psi_i) = L^n(\psi_i) = \lambda_i^n \psi_i, \qquad B_n^*(\phi_i) = \lambda_i^n \phi_i. \tag{18}$$

Then we can drop t_n dependence from (11) and obtain

$$B_{n,\tau_k} = \left[(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i, B_n \right], \tag{19a}$$

$$B_n(\psi_i) = \lambda_i^n \psi_i, \qquad B_n^*(\phi_i) = \lambda_i^n \phi_i, \qquad i = 1, 2, \dots, N, \tag{19b}$$

with the Lax pair given by

$$\Psi_{\tau_k} = \left((B_n)_+^{\frac{k}{n}} + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i \right)(\Psi), \qquad B_n(\Psi) = \lambda^n \Psi.$$

Equation (19) can be regarded as discrete (1+1)-dimensional integrable hierarchy with self-consistent sources. When $n = 2, k = 1$, (19) gives rise to

$$2\Delta\left(f_{0\tau_1} - f_0 f_{0\tau_1} \right) = (\Delta + 2) f_{0\tau_1\tau_1} + \sum_{i=1}^N \left[\Delta^2 \left(f_0 + f_0^{(-1)} - 2 \right) \psi_i \phi_i^{-1} \right.$$
$$\left. + \Delta\left(\psi_i^{(2)} \phi_i - \psi_i \phi_i^{(-2)} \right) + \Delta(\Gamma + 1)\left(\psi_i \phi_i^{(-1)} \right)_{\tau_1} \right] \tag{20a}$$

$$\Delta^2(\psi_i) + \left(f_0 + f_0^{(1)} \right) \Delta(\psi_i) + \left(\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2 \right) \psi_i = \lambda_i^2 \psi_i, \tag{20b}$$

$$\Delta^{*2}(\psi_i) + \Delta^*\left(\left(f_0 + f_0^{(1)} \right) \psi_i \right) + \left(\Delta(f_0) + f_1^{(1)} + f_1 + f_0^2 \right) \psi_i = \lambda_i^2 \phi_i, \tag{20c}$$

which can be transformed to the first type of the Veselov–Shabat equation [20] with self-consistent sources (VSESCS).

3.2. The τ_k-reduction

The τ_k-reduction is given by [10]

$$L^k = B_k + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i.$$

By dropping τ_k dependence from (11), we obtain

$$\left(B_k + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i \right)_{t_n} = \left[\left(B_k + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i \right)_+^{\frac{n}{k}}, B_k + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i \right], \tag{21a}$$

$$\psi_{i,t_n} = \left(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i \right)_+^{\frac{n}{k}} (\psi_i), \tag{21b}$$

$$\phi_{i,t_n} = - \left(B_k + \sum_{i=1}^{N} \psi_i \Delta^{-1} \phi_i \right)_+^{\frac{n}{k}*} (\phi_i), \qquad i = 1, 2, \ldots, N, \tag{21c}$$

which is the k-constrained DKP hierarchy. When $n = 1, k = 2$, (21) leads to

$$2\Delta\left(f_{0t_1} - f_0 f_{0t_1}\right) = (\Delta + 2) f_{0t_1 t_1} + \Delta^2 \sum_{i=1}^{N} \left(\psi_i \phi_i^{(-1)}\right), \tag{22a}$$

$$\psi_{i,t_1} = \Delta(\psi_i) + f_0 \psi_i, \qquad \phi_{i,t_1} = -\Delta^*(\phi_i) - f_0 \phi_i, \qquad i = 1, 2, \ldots, N, \tag{22b}$$

which can be transformed to the second type of VSESCS.

4. Dressing approach for exDKPH

4.1. Dressing approach for the discrete KP hierarchy

We first briefly recall the dressing approach for DKPH [11]. Assume that operator L of DKPH (4) can be written as a dressing form

$$L = W \Delta W^{-1}, \qquad W = \Delta^N + w_1 \Delta^{N-1} + w_2 \Delta^{N-2} + \cdots + w_N. \tag{23}$$

It is known [12] that if W satisfies

$$W_{t_n} = -L_-^n W, \tag{24}$$

then L satisfies (4). It is easy to check the following lemma.

Lemma 2. *If* $h_{t_n} = \Delta^n(h)$, W *satisfies* (24), *then* $\psi = W(h)$ *satisfies* (7), *i.e.*
$$\psi_{t_n} = B_n(\psi). \tag{25}$$

If there are N independent functions $h_1, \ldots, h_N$ solving $W(h) = 0$, i.e. $W(h_i) = 0$, then $w_1, \ldots, w_N$ are completely determined from these h_i, by solving the linear equation:

$$\begin{pmatrix} h_1 & \Delta(h_1) & \cdots & \Delta^{N-1}(h_1) \\ h_2 & \Delta(h_2) & \cdots & \Delta^{N-1}(h_2) \\ \vdots & \vdots & \vdots & \vdots \\ h_N & \Delta(h_N) & \cdots & \Delta^{N-1}(h_N) \end{pmatrix} \begin{pmatrix} w_N \\ w_{N-1} \\ \vdots \\ w_1 \end{pmatrix} = - \begin{pmatrix} \Delta^N(h_1) \\ \Delta^N(h_2) \\ \vdots \\ \Delta^N(h_N) \end{pmatrix}.$$

Then the operator W can be written as

$$W = \frac{1}{Wrd(h_1, \ldots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ \Delta(h_1) & \Delta(h_2) & \cdots & \Delta(h_N) & \Delta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^N(h_1) & \Delta^N(h_2) & \cdots & \Delta^N(h_N) & \Delta^N \end{vmatrix}, \tag{26}$$

where

$$Wrd(h_1, \ldots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ \Delta(h_1) & \Delta(h_2) & \cdots & \Delta(h_N) \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{N-1}(h_1) & \Delta^{N-1}(h_2) & \cdots & \Delta^{N-1}(h_N) \end{vmatrix}.$$

Proposition 2. *Assume that h_i satisfies*

$$h_{i,t_n} = \Delta^n(h_i), \qquad i = 1, \ldots, N \tag{27}$$

W and L are constructed by (26) and (23), then W and L satisfy (24) and (4), respectively.

Proof. Taking partial derivative ∂_{t_n} to the equation $W(h_i) = 0$:

$$W_{t_n}(h_i) + W\Delta^n(h_i) = (W_{t_n} + L_+^n W + L_-^n W)(h_i)$$
$$= (W_{t_n} + L_-^n W)(h_i) = 0, \qquad i = 1, \ldots, N,$$

since $L_-^n W = L^n W - L_+^n W = W\Delta^n - L_+^n W$, $L_-^n W$ is a non-negative difference operator of order $< N$, $W_{t_n} + L_-^n W$ is also of order $< N$. Then according to the difference equation's theory, $W_{t_n} + L_-^n W$ is a zero operator. $\qquad\square$

4.2. Dressing approach for exDKPH

We now generalized the dressing approach to exDKPH (9). We have the following lemma.

Lemma 3. *Under (23), if W satisfies (24) and*

$$W_{\tau_k} = -L_-^k W + \sum_{i=1}^N \psi_i \Delta^{-1} \phi_i W \tag{28}$$

then L satisfies (9a) and (9b).

Proof. It is known that L satisfies (9a). We have

$$L_{\tau_k} = W_{\tau_k}\Delta W^{-1} - W\Delta W^{-1}W_{\tau_k}W^{-1}$$
$$= \left(-L_-^k + \sum_i \psi_i\Delta^{-1}\phi_i\right)L + L\left(L_-^k - \sum_i \psi_i\Delta^{-1}\phi_i\right) = \left[B_k + \sum_{i=1}^N \psi_i\Delta^{-1}\phi_i, L\right]. \qquad\square$$

This dressing operator W is constructed as follows: let $g_i, \bar{g}_i$ satisfy

$$g_{i,t_n} = \Delta^n(g_i), \qquad g_{i,\tau_k} = \Delta^k(g_i) \tag{29a}$$

$$\bar{g}_{i,t_n} = \Delta^n(\bar{g}_i), \qquad \bar{g}_{i,\tau_k} = \Delta^k(\bar{g}_i), \qquad i = 1, \ldots, N. \tag{29b}$$

And let h_i be the linear combination of g_i and $\bar{g}_i$,

$$h_i = g_i + \alpha_i(\tau_k)\bar{g}_i \qquad i = 1, \ldots, N, \tag{30}$$

with the coefficient α_i being a differentiable function of τ_k. Suppose $h_1, \ldots, h_N$ are still linearly independent.

Define

$$\psi_i = -\dot{\alpha}_i W(\bar{g}_i), \qquad \phi_i = (-1)^{N-i}\frac{\mathrm{Wrd}(\Gamma h_1, \ldots, \hat{\Gamma h_i}, \ldots, \Gamma h_N)}{\mathrm{Wrd}(\Gamma h_1, \ldots, \Gamma h_N)}, \qquad i = 1, \ldots, N, \tag{31}$$

where the hat $\hat{\ }$ means rule out this term from the discrete Wronskian determinant, $\dot{\alpha}_i = \frac{\mathrm{d}\alpha_i}{\mathrm{d}\tau_k}$. We have the following proposition.

Proposition 3. *Let W be defined by (26) and (30), $L = W\Delta W^{-1}$, ψ_i and ϕ_i be given by (31), then W, L, ψ_i, ϕ_i satisfy (24), (28) and exDKPH (9).*

To prove it, we need several lemmas under the above assumptions. The first one is the following:

Lemma 4. *(The discrete version of Oevel and Strampp's lemma [18])*

$$W^{-1} = \sum_{i=1}^{N} h_i \Delta^{-1} \phi_i.$$

Proof. Note that $\phi_1, \ldots, \phi_N$ defined in (31) satisfy the linear equation

$$\sum_{i=1}^{N} \Delta^j (\Gamma h_i) \cdot \phi_i = \delta_{j,N-1}, \qquad j = 0, 1, \ldots, N-1, \qquad (32)$$

where $\delta_{j,N-1}$ is the *Kronecker's delta* symbol. Using properties $f\Delta^{-1} = \sum_{j \geqslant 0} \Delta^{-j-1} \Delta^j (\Gamma f)$, we have

$$\sum_{i=1}^{N} h_i \Delta^{-1} \phi_i = \sum_{i=1}^{N} \sum_{j=0}^{\infty} \Delta^{-j-1} \Delta^j (\Gamma(h_i)) \cdot \phi_i = \sum_{j=0}^{\infty} \Delta^{-j-1} \sum_{i=1}^{N} \Delta^j (\Gamma(h_i)) \cdot \phi_i$$

$$= \sum_{j=0}^{N-1} \Delta^{-j-1} \delta_{j,N-1} + \sum_{j=N}^{\infty} \Delta^{-j-1} \sum_{i=1}^{N} \Delta^j (\Gamma(h_i)) \cdot \phi_i = \Delta^{-N} + O(\Delta^{-N-1}).$$

So we have

$$W \sum_i h_i \Delta^{-1} \phi_i = 1 + \left(W \sum_i h_i \Delta^{-1} \phi_i \right)_- = 1 + \sum_i W(h_i) \Delta^{-1} \phi_i = 1. \qquad (33)$$

This completes the proof. $\qquad\qquad\square$

Lemma 5. $W^*(\phi_i) = 0$, *for* $i = 1, \ldots, N$.

Proof. Lemma 1 implies that

$$(\Delta^{-1} \phi_i W)_- = \Delta^{-1} W^*(\phi_i). \qquad (34)$$

Using lemma 4 and (10*a*), we have

$$0 = (\Delta^j W^{-1} W)_{-1} = \left(\Delta^j \sum_{i=1}^{N} h_i \Delta^{-1} \phi_i W \right)_- = \left(\sum_{i=1}^{N} \Delta^j (h_i) \Delta^{-1} \phi_i W \right)_-$$

$$= \sum_{i=1}^{N} \Delta^j (h_i) \Delta^{-1} W^*(\phi_i), \qquad j = 0, \ldots, N-1.$$

Solving the equations with respect to $\Delta^{-1} W^*(\phi_i)$, we find $\Delta^{-1} W^*(\phi_i) = 0$. This implies $W^*(\phi_i) = 0$. $\qquad\qquad\square$

Lemma 6. *The operator* $\Delta^{-1} \phi_i W$ *is a non-negative difference operator and*

$$(\Delta^{-1} \phi_i W)(h_j) = \delta_{ij}, \qquad 1 \leqslant i, \quad j \leqslant N. \qquad (35)$$

Proof. Lemma 5 and (34) imply that $\Delta^{-1} \phi_i W$ is a non-negative difference operator. We define functions $c_{ij} = (\Delta^{-1} \phi_i W)(h_j)$, then $\Delta(c_{ij}) = \phi_i W(h_j) = 0$, which means c_{ij} does not depend on the discrete variable n. From lemma 4, we find that

$$\sum_{i=1}^{N} \Delta^k (h_i) c_{ij} = \Delta^k \left(\sum_i (h_i \Delta^{-1} \phi_i W)(h_j) \right) = \Delta^k (W^{-1} W)(h_j) = \Delta^k (h_j),$$

so $c_{ij} = \delta_{ij}$. $\qquad\qquad\square$

Proof of Proposition 3. The proof of (24) is analogous to the proof in the previous section. For (28), taking ∂_{τ_k} to the identity $W(h_i) = 0$, using (29), (30), the definition (31) and lemma 6, we find

$$0 = (W_{\tau_k})(h_i) + (W\Delta^k)(h_i) + \dot{\alpha}_i W(\bar{g}_i) = (W_{\tau_k})(h_i) + (L^k W)(h_i) - \sum_{j=1}^{N} \psi_j \delta_{ji}$$

$$= \left(W_{\tau_k} + L_-^k W - \sum_{j=1}^{N} \psi_j \Delta^{-1} \phi_j W\right)(h_i).$$

Since the non-negative difference operator acting on h_i in the last expression has degree $< N$, it cannot annihilate N independent functions unless the operator itself vanishes. Hence (28) is proved. Then lemma 3 leads to (9*b*). The first equation in (9*c*) is easy to be verified by a direct calculation, so it remains to prove the second equation in (9*c*). First, we see that

$$(W^{-1})_{t_n} = -W^{-1} W_{t_n} W^{-1} = W^{-1}(L^n - B_n) = \Delta^n W^{-1} - W^{-1} B_n.$$

If we substitute $W^{-1} = \sum h_i \Delta^{-1} \phi_i$ into this equality at both ends, we have

$$(W^{-1})_{t_n} = \sum \Delta^n(h_i)\Delta^{-1}\phi_i + \sum h_i \Delta^{-1}\phi_{i,t_n}$$

$$= (\Delta^n W^{-1} - W^{-1} B_n)_- = \sum \Delta^n(h_i)\Delta^{-1}\phi_i - \sum h_i \Delta^{-1} B_n^*(\phi_i).$$

Then $\sum h_i \Delta^{-1}\phi_{i,t_n} = -\sum h_i \Delta^{-1} B_n^*(\phi_i)$ implies that (9*c*) holds.

5. *N*-soliton solutions for exDKPH

Using proposition 3, we can find solutions to every equations in the exDKPH (9). Let us illustrate it by solving (13) and (15). For (13), let $\delta_i = e^{\lambda_i} - 1$, $\kappa_i = e^{\mu_i} - 1$, we take the solution of (29) as follows:

$$g_i := \exp\left(l\lambda_i + \delta_i t_1 + \delta_i^2 \tau_2\right) = e^{\xi_i}, \qquad \bar{g}_i := \exp\left(l\mu_i + \kappa_i t_1 + \kappa_i^2 \tau_2\right) = e^{\eta_i}$$

$$h_i := g_i + \alpha_i(\tau_2)\bar{g}_i = 2\sqrt{\alpha_i}\exp\left(\frac{\xi_i + \eta_i}{2}\right)\cosh(\Omega_i), \qquad \Omega_i = \frac{1}{2}(\xi_i - \eta_i - \ln\alpha_i). \tag{36}$$

Since $L = W\Delta W^{-1} = \Delta + f_0 + f_1\Delta^{-1} + \cdots$, we have

$$f_0 = \text{Res}_\Delta(W\Delta W^{-1}\Delta^{-1}), \tag{37}$$

where W is given by (26) and (36), then f_0, ψ_i and ϕ_i given by (31) give rise to the *N*-soliton solution for (13).

For example, we obtain a 1-soliton solution for (13) with $N = 1$ as follows:

$$f_0 = \exp\left(\frac{\lambda_1 + \mu_1}{2}\right)\left(\frac{\cosh(\Omega_1 + 2\theta_1)}{\cosh(\Omega_1 + \theta_1)} - \frac{\cosh(\Omega_1 + \theta_1)}{\cosh\Omega_1}\right), \qquad \theta_1 = \frac{\lambda_1 - \mu_1}{2}$$

$$\psi_1 = -\frac{\mathrm{d}\sqrt{\alpha_1}}{\mathrm{d}\tau_2}(e^{\mu_1 - \lambda_1})\exp\frac{\xi_1 + \eta_1}{2}\,\text{sech}\,\Omega_1, \qquad \phi_1 = \frac{e^{-(\lambda_1 + \mu_1)/2}\exp\left(-\frac{\xi_1 + \eta_1}{2}\right)}{2\sqrt{\alpha_1}}\text{sech}(\Omega_1 + \theta_1).$$

The 2-soliton solution of (13) with $N = 2$ is given by

$$f_0 = -\Delta(w_1) = (e^{\lambda_1} + e^{\lambda_2})\Delta\left(\frac{v_1}{v}\right),$$

$$\psi_1 = -\frac{\dot{\alpha}_1}{v}\left(1 + \alpha_2 \frac{(e^{\mu_2} - e^{\lambda_1})(e^{\mu_1} - e^{\mu_2})}{(e^{\lambda_2} - e^{\lambda_1})(e^{\mu_1} - e^{\lambda_2})}e^{\chi_2}\right)(e^{\mu_1} - e^{\lambda_1})(e^{\mu_1} - e^{\lambda_2})\,e^{\eta_1},$$

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$$\psi_2 = -\frac{\dot{\alpha}_2}{v}\left(1 + \alpha_1 \frac{(e^{\mu_1}-e^{\lambda_2})(e^{\mu_1}-e^{\mu_2})}{(e^{\lambda_2}-e^{\lambda_1})(e^{\mu_2}-e^{\lambda_1})}e^{\chi_2}\right)(e^{\mu_2}-e^{\lambda_2})(e^{\mu_2}-e^{\lambda_1})\,e^{\eta_2},$$

$$\phi_1 = \Gamma\left(\frac{1+\alpha_2\,e^{\chi_2}}{(e^{\lambda_1}-e^{\lambda_2})v}e^{-\xi_1}\right), \qquad \phi_2 = \Gamma\left(\frac{1+\alpha_1 e^{\chi_1}}{(e^{\lambda_2}-e^{\lambda_1})v}e^{-\xi_2}\right),$$

with

$$v = 1 + \alpha_1 \frac{e^{\lambda_2}-e^{\mu_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_1} + \alpha_2 \frac{e^{\mu_2}-e^{\lambda_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_2} + \alpha_1\alpha_2\frac{e^{\mu_2}-e^{\mu_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_1+\chi_2},$$

$$v_1 = 1 + \alpha_1 \frac{e^{2\lambda_2}-e^{2\mu_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_1} + \alpha_2 \frac{e^{2\mu_2}-e^{2\lambda_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_2} + \alpha_1\alpha_2\frac{e^{2\mu_2}-e^{2\mu_1}}{e^{\lambda_2}-e^{\lambda_1}}e^{\chi_1+\chi_2}.$$

It can be shown that the interaction between the two solutions is elastic.

For (15), we take the solution of (29) as follows:

$$g_i := \exp\left(l\lambda_i + \delta_i\tau_1 + \delta_i^2 t_2\right) = e^{\xi_i}, \qquad \bar{g}_i := \exp\left(l\mu_i + \kappa_i\tau_1 + \kappa_i^2 t_2\right) = e^{\eta_i}$$

$$h_i := g_i + \alpha_i(\tau_1)\bar{g}_i = 2\sqrt{\alpha_i}\exp\left(\frac{\xi_i+\eta_i}{2}\right)\cosh(\Omega_i).$$

Then

$$f_0 = \operatorname{Res}_\Delta(W\Delta W^{-1}\Delta^{-1}), \qquad f_1 = \operatorname{Res}_\Delta(W\Delta W^{-1})$$

together with ψ_i and ϕ_i given by (31) presents the *N*-soliton solution for (15).

Acknowledgments

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